where sgn = 1 if $r > s$, and sgn = $(-1)^{p+q}$ if $r < s$; and x_{0r} and x_{0s} are the x values of the centers of slots r and s respectively.

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A Uniform Asymptotic Expansion for the Green's Functions Used in Microstrip Calculations

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Abstract —**A uniform asymptotic approximation is developed in the limit of small substrate thickness for the Green's functions used in microstrip-type problems. The approximation is valid for a single-layer substrate. The expansions agree with near and far-field results previously published in the literature. Comparison of the approximation is made with numerical evaluations of the exact integral solution available for the problem.**

I. Introduction

There is currently a great deal of interest in the numerical computation of microstrip circuit parameters. One of the most popular approaches has been the use of moment method techniques. For example, Gardiol and Mosig [1], [2] have developed algorithms for a complete moment method solution. (See also Mosig [3].) These methods invariably require a knowledge of various Green's functions in order for the integral equation to be properly formulated. Unfortunately, the Green's functions are of the Sommerfeld integral type and are not known in terms of simple functions. The integrals that must be evaluated are slow to converge and exhibit nearly singular behavior.

Various researchers have developed approximations to these integrals in the limit of high and low frequencies and for near and far fields. In this paper, I will develop an approximation which is uniformly valid for all distances from the source in the limit where the substrate region is thin, which is typically the case of interest for microstrip problems. The advantages of having a uniform asymptotic expansion available are that it can lead to much quicker evaluation of the matrix elements in the discrete integral equation, and it gives the researcher a much better feel for how the fields behave. In addition, it can serve as the starting point for more accurate approximations if such approximations are needed. There are a number of asymptotic expansions available in the literature for Green's functions of this type. Unfortunately, none of them is valid for all distances from the source. Approximations exist for the quasi-static region: $|k_0\rho| \ll 1$, where ρ is the radial distance from the dipole source and k_0 is the free-space wavenumber [3]. Sophisticated approximations exist for $|k\rho| > 1$, where k is the wavenumber of the substrate [4]. The main result presented in this paper is to show how such expressions can be combined to make a uniform approximation for all distances, ρ, if the substrate is electrically thin. The method is based on work carried out by Wu and King [6], [7]; their work examines the two-layer semi-infinite problem. In addition, King [8] has carried out an analysis of the microstrip case for thin substrates when the distance from the source is much greater than a substrate thickness.

The approximations are derived in the next section. The results are compared with numerical evaluations of the exact Green's functions in the third part of the paper.

II. Derivation of the Formulas

In this section, the uniform asymptotic expansions are derived. The expansions are developed for the scalar and vector potentials rather than for the electromagnetic fields. This is done because they have slightly simpler integrals to evaluate, and because it is useful to formulate moment method numerical equations in terms of potentials. It is possible to work directly from the fields if one wishes.

We use the formulas given in Mosig [3] as the basis of the expansions. It is shown that the electric and magnetic fields can be written as

$$\vec{E} = -\nabla V - j\omega\vec{A}$$
$$\vec{B} = \nabla \times \vec{A} \tag{1}$$

where $\vec{E}$ and $\vec{B}$ are the electric field and magnetic flux density vectors, $\vec{A}$ is the vector potential, and V is the scalar potential, defined in the usual way. The equations are written in the frequency domain with an $(\exp(j\omega t))$ time dependence assumed. MKS units are used. A unit strength time-harmonic electric dipole is placed on the interface between the air and substrate regions and is oriented in the x direction (see Fig. 1). The coordinate system is chosen so that the z axis is vertical. The interface between the two media is on the $z = 0$ plane, and the perfectly conducting ground plane is on the $z = -h$ plane. The substrate thickness is, therefore, h. Quantities which refer to the upper region have a subscript 0. It is shown in [3] that the fields from the dipole can be completely determined if the Green's functions are known: $G_{xx}^A(\vec{\rho},0)$ and $G_V(\vec{\rho},0)$. $G_{xx}^A(\vec{\rho},0)$ is the xx component of the dyadic Green's function for the vector potential $\vec{A}$ arising from a unit electric dipole in the x direction at the origin. G_V is the Green's function for the scalar potential V. To get the actual potentials for a given charge and

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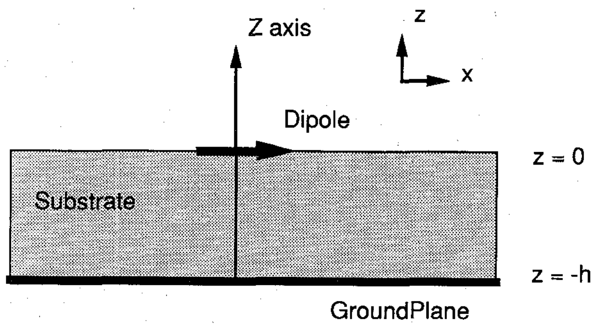

Fig. 1. Geometry of the problem. The dipole is located on the interface between the substrate and the air.

current distribution, G_V is integrated with the charge, and G_{xx}^A is integrated with the current. The moment method equations are therefore formulated with both charge and current cells. This type of formulation has certain numerical advantages, although charge conservation must be enforced numerically.

It is natural to define a cylindrical coordinate system with $\vec{\rho}$ being a radial vector going out from the origin. By symmetry, the Green's functions will depend only on ρ and z, not the angular coordinate, ϕ. In this paper expressions are developed only for the Green's functions at the interface between the two media, $z = 0$. This is because in microstrip applications the metal circuit is on the substrate's surface.

The exact expressions for the Green's functions are [3]

$$G_A^{xx}(\vec{\rho},0) \equiv A_x(\vec{\rho}) = \frac{\mu_0}{2\pi}\int_0^\infty dk_\rho J_0(k_\rho \rho)\frac{k_\rho}{D_{\mathrm{TE}}}$$

$$G_V(\vec{\rho},0) \equiv V(\vec{\rho}) = \frac{1}{2\pi\epsilon_0}\int_0^\infty dk_\rho J_0(k_\rho\rho)k_\rho \frac{u_0 + u\tanh(uh)}{D_{\mathrm{TE}}D_{\mathrm{TM}}} \tag{2}$$

where

$$D_{\mathrm{TE}} = u_0 + u\coth(uh)$$

$$D_{\mathrm{TM}} = \epsilon_r u_0 + u\tanh(uh) \tag{3}$$

μ_0 being the free-space permeability; magnetic materials are excluded from this discussion. The quantity ϵ_r is the relative permittivity of the substrate region, J_0 is a Bessel function of order zero, and u_0 and u are two functions of the integration variable k_ρ:

$$u_0 = \sqrt{k_\rho^2 - k_0^2}$$

$$u = \sqrt{k_\rho^2 - k^2}. \tag{4}$$

Here k_0 is the free-space wavenumber, and $k = \sqrt{\epsilon_r}k_0$ is the wavenumber of the substrate region. It is assumed that the substrate is lossless. The functions are given unique definitions in the complex k_ρ plane by requiring that $\mathrm{Re}(u) > 0$ and $\mathrm{Re}(u_0) > 0$ for all k_ρ. There are no branch cuts for u as the integrands are even in u. (See Felsen and Marcuvitz [4] for details concerning the choice of cuts.) Integrals of the type discussed in this paper are known in the literature as Sommerfeld integrals. These integrals typically do not converge when the source and the point of observation are located in the same plane, because of their behavior as $k_\rho \to \infty$. The integrals can be given an unambiguous meaning by defining them to be the limit as the source point goes to the interface.

The asymptotic expansions for the Green's functions are now developed. The goal is to obtain formulas that are valid in the limit of small substrate thickness: $|k_0 h| \ll 1$. This condition is usually realized in practical microstrip structures; if it is not, there will be undesirable surface waves generated which will adversely affect the circuit's performance. By a uniform expansion is meant an expansion that is valid for all values of radial distance, ρ. Such an expansion has an advantage over more conventional expansions in that it is not necessary to worry about when to switch from one approximation to another.

The first step in deriving a uniform asymptotic expression for the potentials is to develop expansions for A_x and V for $\rho \gg h$ in the limit $|k_0 h| \ll 1$. This is carried out in a straightforward way; the integrands are expanded for small $|k_0 h|$. The resulting integrals are known analytically. It is found that

$$\frac{2\pi}{\mu_0}A_x \approx \frac{h^2 e^{-jk_0\rho}}{\rho^2}\left[k_0 j + \frac{1}{\rho}\right]$$

$$2\pi\epsilon_0 V \approx \frac{h^2}{\epsilon_r^2}e^{-jk_0\rho}\left[-(\epsilon_r - 1)^2\frac{k_0^2}{\rho} + \frac{1}{\rho^2}\left(\frac{1}{\rho} + jk_0\right)\right]. \tag{5}$$

These expressions are valid for $O(|k_0 h|^2)$. The next order correction is $O(|k_0 h|^4)$.

Equation (5) for V is not correct in its present form. It predicts that V will decay as $1/\rho$ for large distances. This cannot be true. The exact integrand for V contains a pole at $k_\rho = k_{\mathrm{pole}}$. This pole occurs where $D_{\mathrm{TM}}(k_{\mathrm{pole}}) = 0$. The pole, however, is lost when the approximation leading to (5) is carried out. Two modifications to the expression must be made. First of all, the surface wave contribution from the pole must be included. Second, the effect of a branch cut near the pole must be included. The pole near the branch cut can be examined by looking at a simplified integral which can be analytically evaluated and yet retains the essential behavior near the pole. The integral used is

$$I = h\int_0^\infty dk_\rho k_\rho J_0(k_\rho\rho)\left[\frac{u_0 + u^2 h}{(u_0 h + 1)(\epsilon_r u_0 + u^2 h)} - \frac{1}{\epsilon_r} - \frac{h}{\epsilon_r u_0}\left[(k_0^2 - k^2) - \frac{u^2}{\epsilon_r}\right]\right]. \tag{6}$$

The last two terms of the integrand can be evaluated exactly; they give the approximation for V in (6) (with a minus sign). The first term in (6) is a simplified version of the exact integrand, but one which keeps the pole. The total integrand in (6) is significant only near the pole. It can therefore be simplified. For small $|k_0 h|$, the pole's position, k_{pole}, is given by

$$k_{\mathrm{pole}} \approx k_0 + \frac{(k_0^2 - k^2)^2 h^2}{2\epsilon_r^2 k_0} + O((kh)^4). \tag{7}$$

The integrand is expanded around the pole. A new variable, τ, is defined by

$$k_\rho = k_0(1 + m\tau) \tag{8}$$

where m is given by

$$m = \frac{h^2(k_0^2 - k^2)^2}{2\epsilon_r^2 k_0^2}. \tag{9}$$

Notice that $|m| \ll 1$ by assumption. This change of variables gives approximate values of u and u_0 of

$$u \approx \sqrt{k_0^2 - k^2}$$
$$u_0 \approx \sqrt{2m\tau}\, k_0. \tag{10}$$

The resulting integral in τ can be evaluated analytically after the Bessel function is converted to its far-field form. The procedure is very similar to that carried out in the appendix of [6]. The resulting expression for I is

$$I = A(1-\epsilon_r) j 2\pi e^{-j\alpha} \operatorname{erfc}\left(\sqrt{\alpha}\, e^{-j\pi/4}\right) \tag{11}$$

where A and α are given by

$$A = \frac{hk_0^2 m}{2\epsilon_r}\sqrt{\frac{2}{\pi k_0 \rho}}\, e^{-jk_0\rho} e^{j\pi/4}$$
$$\alpha = k_0 \rho m \tag{12}$$

and where "erfc" is the complementary error function [9],

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty d\zeta\, e^{-\zeta^2}. \tag{13}$$

The surface wave term and the integral I must be added to the expression for V in (5) to get a valid approximation for $\rho \gg h$. It should be noted that when ρ is very large, the asymptotic expansion for large argument for the complementary error function may be used. The leading order term will cancel the $1/\rho$ term in (5). The first nonzero contribution will therefore be $1/\rho^2$, as expected. The surface wave term will eventually dominate, as it goes as $1/\sqrt{\rho}$ for large ρ.

The approximations discussed so far are valid only for $\rho \gg h$, to $O(|k_0 h|^2)$. It is not known how to get an approximation for $\rho \ll h$. However, there are known quasi-static approximations for which $|k_0 \rho| \ll 1$ [3]. These expressions are valid if ρ is small enough. What is not clear is whether the quasi-static expressions and the $\rho \gg h$ expression have any common region of validity. The quasi-static approximations for A_x and V are

$$\frac{4\pi}{\mu_0} A_x = \frac{e^{-jk_0 R_0}}{R_0} - \frac{e^{-jk_0 R_1}}{R_1}$$
$$4\pi\epsilon_0 V = (1-\eta)\left[\frac{e^{-jk_0 R_0}}{R_0} - (1+\eta)\sum_{i=1}^{\infty} (-\eta)^{i-1} \frac{e^{-jk_0 R_i}}{R_i}\right] \tag{14}$$

where

$$R_i^2 = \rho^2 + (2ih)^2 \tag{15}$$

and

$$\eta = \frac{\epsilon_r - 1}{\epsilon_r + 1}. \tag{16}$$

Equations (14) are expanded for $\rho \gg h$. The expression for A_x is found to agree with that in (5) to $O(|k_0 h|^2)$. The expression for V in (14) can be expanded for $\rho \gg h$ using contour integration. The series is rewritten as an integral:

$$\sum_{i=1}^{\infty} (-1)^i \frac{\eta^i}{R_i} e^{-jk_0 R_i} = \frac{1}{2j} \int d\lambda \frac{\eta^\lambda}{\sin(\pi\lambda)} \frac{1}{\sqrt{\rho^2 + 4h^2\lambda^2}} e^{-jk_0\sqrt{\rho^2+4h^2\lambda^2}} \tag{17}$$

where the contour goes counterclockwise around the positive real axis. The contour is then deformed to lie on the line $\lambda = 1/2$. The integrand is expanded for small h. It is found that

$$\lim_{|h/\rho| \to 0} \sum_{i=1}^{\infty} (-1)^i \frac{\eta^i}{R_i} e^{-jk_0 R_i} \approx -\frac{1}{\rho}\left(\frac{\eta}{1+\eta}\right) e^{-jk_0\rho} + \left(\frac{2jh^2}{\rho}\right)\left(\frac{\eta(1-\eta)}{(1+\eta)^3}\right) e^{-jk_0\rho}\left(\frac{k_0}{\rho} - \frac{j}{\rho^2}\right). \tag{18}$$

The $1/\rho$ term cancels the $1/\rho$ term in (14). The $1/\rho^2$ and $1/\rho^3$ terms match up with the corresponding terms in (5). The quasi-static expression thereby matches up with the approximation for $\rho \gg h$ to $O(|k_0 h|^2)$, and a uniformly asymptotic expression can be written. It is found that

$$2\pi\epsilon_0 V = -\frac{h^2 k_0^2 (\epsilon_r - 1)^2}{\epsilon_r^2 \rho} e^{-jk_0\rho} + \frac{1-\eta}{2}\left[\frac{1}{\rho} e^{-jk_0\rho} - (1+\eta)\sum_{i=1}^{\infty} (-\eta)^{i-1} \frac{1}{R_i} e^{-jk_0 R_i}\right] + A(1-\epsilon_r) j 2\pi e^{-j\alpha} \operatorname{erfc}\left(\sqrt{\alpha}\, e^{-j\pi/4}\right) + \text{surface wave term.} \tag{19}$$

The surface wave term must be included for a complete description of V. After some manipulation, it can be shown that the surface wave contribution is as follows:

$$\text{surface wave term} = \frac{\pi j H_0^{(2)}(k_{\text{pole}}\rho)\, u_0^2 \epsilon_r}{k_{\text{pole}}^2\left[\epsilon_r/u_0 + 1/u\left(\tanh(uh) + uh\,\operatorname{sech}^2(uh)\right)\right]}. \tag{20}$$

The expression for the surface wave can be simplified for small substrate thickness. Expressions for k_{pole}, given by (7), and for u_0, given by (10), are substituted into (20). It is found that

$$\text{surfave wave term} \approx \pi j H_0^{(2)}(k_{\text{pole}}\rho) \frac{|k_0^2 - k^2|^3 h^3}{\epsilon_r^3 k_0^2} + O(|kh|^5). \tag{21}$$

The most time consuming part of the calculation of (19) is that of summing the series. In practice, it is found that the approximation given in (18) can be safely used when $|\rho| \geq 10|h|$ for ϵ_r in the range of 12.9. The substitution of (18) into (19) greatly speeds up the calculation.

III. Comparison with Numerical Results

In this section, the approximate expressions are compared with numerical evaluation of the exact integrals. The numerical evaluations were carried out using methods similar to those used by Mosig [3]. The expression for A_x is not examined here as it is identical to the quasi-static expression previously discussed in the literature. For example, a number of curves comparing this approximation with a numerical evaluation of the exact integrals are shown in [3].

In Fig. 2 are shown plots of the magnitude of V versus radial distance for three different substrate thicknesses. Both distance and thickness are normalized to the free-space wavenumber, k_0. Values of $k_0 h$ of 0.05, 0.15, and 0.25 are shown. The substrate's relative permittivity is chosen to be that of GaAs, $\epsilon_r = 12.9$. Decibel scales are used for both the magnitude of V ($20\log_{10}|V|$)

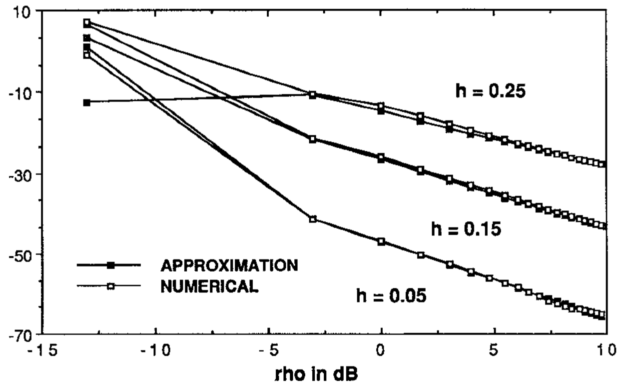

Fig. 2. Comparison of numerical calculations and the analytical approximation for $|V|$. Distance is in dB: $\log_{10} k_0\rho$. Results are shown for $k_0h = 0.05$, 0.15, and 0.25.

and $k_0\rho(20\log_{10}|k_0\rho|)$. The agreement between the numerical calculations and analytical approximation gets progressively worse as the substrate thickness is increased. The largest error occurs for the value of $k_0\rho$ near -13 dB. The reason is that the approximation for V in (19) has two $1/\rho$ terms in it. These terms dominate all others for small values of ρ. The first of these terms is due to the approximate expression for $\rho \gg h$. The second is due to the direct term in the quasi-static approximation. The direct term should dominate for distances close to the dipole, as can be seen from the numerical results. The first $1/\rho$ term is assumed to be negligible compared with the direct term, as it is smaller by a factor $|k_0h|^2$. It is seen from (19) that this is true when

$$|kh_0|^2 \ll \frac{\epsilon_r^2}{(\epsilon_r-1)^2(\epsilon_r+1)}. \tag{22}$$

This condition is violated for values of $k_0h = 0.25$. The solution to this problem is that one should not include the first $1/\rho$ term for small values of ρ. (The surface wave term should not be included either. This, however, gives negligible corrections, as the surface wave will go as $\log|\rho|$ for small values of ρ, which is much smaller than $1/\rho$.)

The maximum relative error for the points plotted for $k_0h = 0.05$ is about 1.5% if the $1/\rho$ term mentioned above is left out for $\rho < h$. Similarly, the maximum relative error for $k_0h = 0.15$ is approximately 6%, and for $k_0h = 0.25$ is approximately 11% for the points plotted.

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Efficient Computation of the Free-Space Periodic Green's Function

Surendra Singh and Ritu Singh

Abstract —**The application of Shanks's transform is shown to improve the convergence of the series representing the doubly infinite free-space periodic Green's function. Higher order Shanks transforms are computed via Wynn's ϵ algorithm. Numerical results confirm that a dramatic improvement in the convergence rate is obtained for the "on-plane" case, in which the series converges extremely slowly. In certain instances, the computation time can be reduced by as much as a factor of a few thousands. A relative error measure versus the number of terms taken in the series is plotted for various values of a convergence factor as the observation point is varied within a unit cell. Computation times are also provided.**

I. Introduction

The problem of determining the radiation or scattering from a periodic array geometry is formulated in terms of an integral equation. The integral equation is solved numerically via the method of moments. In the moment method solution the unknown surface current or field is expanded either in terms of entire domain basis functions at the expense of generality or in terms of subdomain basis functions at the expense of higher computation cost. In order to achieve the degree of generality required in developing general-purpose computer codes, it is necessary to employ subsectionally defined basis functions. This requires repeated computations of the free-space periodic Green's function. The Green's function for a two-dimensional periodic array (of point sources of radiating elements or conducting strips) is represented in terms of a doubly infinite series. This series converges extremely slowly as the observation point approaches the source plane. In the moment method solution for the current distribution on the radiator in the reference cell of a two-dimensional infinite periodic array of radiating elements, the observation point lies in the plane of the array. This case is referred to as the on-plane case, and the series has the slowest convergence rate. In comparison with other methods that make use of Kummer's transform to accelerate the conver-

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